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Calculating the Present Value
of Riskless Cash Flows

by
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#1348-82

September 1982

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Calculating the Present Value of Riskless Cash Flows

This note summarizes the answer to a very simple question. The question is:

Is it appropriate for a firm that realizes the full value of its tax shields to calculate the present value of riskless after-tax cash flows by discounting at the riskless interest rate or at the after-corporate tax riskless interest rate?

I asked numerous colleagues this question. Each of them thought the answer was obvious. However, approximately half of them thought the obvious answer was the before tax interest rate, whereas the other half believed the answer to be the after-tax interest rate.

Introductory finance textbooks also seem to disagree on the answer. While the textbooks do not contain an answer to this specific question, Brealey and Myers (1981) imply that the answer is to discount the cash flows at the before tax rate and add the present value of the interest tax shields, whereas Weston and Brigham (1981) imply that the after-tax riskless rate should be used to discount the cash flows. Disagreement is also present in the literature on leasing. Myers, Dill and Bautista (1976) argue that leasing displaces debt and thus the riskless cash flows from the lease should be discounted at the after-tax interest rate. In contrast, Gordon (1974) concludes "a corporation should use the risk free interest rate without deducting the corporate income

tax in arriving at the present value of an after-tax risk free cash flow."
(page 246).

In this note I summarize the arguments for using the before and after tax discount rate and prove two propositions. The first proposition is that the present value of a stream of after tax riskless cash flows can be determined by discounting them at the after-tax discount rate. The second proposition shows that the present value of after tax riskless interest cash flows can be determined by discounting them at the before tax interest rate and adding the present value of the interest tax shields which would be realized if the firm borrowed the present value of the cash flows.

Before presenting the arguments for using the alternative rates, it is important to recognize that personal taxes are irrelevant for the choice of discount rates. Firms take the prices of securities as given. These prices rationally incorporate the investors' tax liability associated with the security. The riskless interest rate, therefore, reflects personal tax rates. This irrelevance of personal tax rates is fortunate since firms would have difficulty measuring the marginal tax rates of investors at any point in time and complications arise in a multiperiod context in which the population of investors could change over time.

The argument for using the before tax rate can be summarized simply. Investors can buy riskless assets directly in the form of Treasury bills or money market funds. Their pre-tax return from these investments is the riskless rate of return - not the riskless rate net of corporate taxes. A firm, therefore, must earn at least the before tax riskless rate after tax to compensate investors for the opportunity cost of capital. Gordon (1974) applies this reasoning to an example in which the firm's only asset is \$M of perpetual bonds which yield taxable income of \$rM in each period. Using the

before-tax interest rate, r , to discount the cash flows, the value of the firm, V , is $V = (1 - \tau)rM/r = (1 - \tau)M$ where τ is the corporate tax rate. Gordon notes that this formulation suggests an arbitrage opportunity: the firm can be purchased for $(1 - \tau)M$, liquidated for $\$M$ and an arbitrage gain of $\$ \tau M$ would be realized. This arbitrage opportunity is eliminated if the after-tax riskless cash flows are discounted at the after-tax discount rate. Equivalently, the firm could offset its interest income by issuing a perpetual bond which pays interest of rM each period. This transaction eliminates all future cash flows and the proceeds of the debt are by construction $\$M$ and equal to the value of the firm.¹ Thus, using the investors' opportunity cost of capital to value riskless cash flows is inappropriate when the firm can offset the riskless cash flows with riskless debt.²

Another answer which focuses on the investors' opportunity cost of capital is that the proper discount rate is determined in the stock market and proxied by the expected return on a zero beta asset since the cash flows are assumed to be riskless. This view raises two concerns immediately. First, why is the

¹The analysis assumes there is no tax disadvantage from corporate borrowing. A tax disadvantage could occur, for example, if there was no personal tax liability on equity income and the personal tax rate for all investors on interest income exceeded the corporate tax rate. Corporate borrowing would not occur since investors would prefer to lever themselves to realize the larger tax shields. Thus, investors would be willing to pay more than M for an equity issue with riskless dividend payments of rM in perpetuity and the value of the firm would exceed M . In general, if there was a tax disadvantage to borrowing, firms could engage in tax arbitrage by purchasing debt and issuing equity claims on the debt. For a more detailed discussion of the effect of personal and corporate taxes on debt tax shields see Miller (1977).

²Henderson (1976) shows that Gordon's (1974) proof is only correct for a firm which is entirely equity financed.

expected rate of return on a zero beta portfolio, with all of the measurement difficulties, superior to the riskless return observed in the bond market? Second, it isn't obvious whether the expected rate of return on a zero beta portfolio is a before or after-tax return. It is an after-tax return in the sense that it is determined by changes in the capitalized value of after-tax cash flows and dividend payments. But it is a before tax return to the corporation that owns the security and pays taxes on the dividends and realized capital gains.

Discounting after-tax cash flows by after-tax interest rates is suggested by the definition of present values. The present value of a riskless stream of cash flows is simply the amount of money that has to be invested in a riskless asset today to exactly duplicate the stream of cash flows after taxes. The most obvious riskless investment is a government bond. The firm is required to pay ordinary income taxes on its interest income from the bond and so the after-tax cash flows from the bond should equal the after-tax stream of cash flows that are being replicated. The after-tax cash flows from the bond discounted at the after-tax rate equals the amount invested. Thus the present value of the after-tax cash flow is determined by discounting them at the after-tax rate.

A related argument for discounting by the after-tax discount rate focuses on the firm's opportunity cost of capital. The firm could obtain financing for a riskless stream of cash flows by borrowing at the riskless rate. Since the interest payments are tax deductible, the firm's cost of capital is the after-tax rate. This implies riskless projects should be evaluated by discounting the project's after-tax cash flows at the after-tax rate.

While each of these arguments appears to have some merit, those suggesting the use of before tax riskless rates are wrong or at least incomplete. In

what follows I present a simple arbitrage proof to show that the present value of a riskless stream of after-tax cash flows is determined by discounting them at the after corporate tax interest rate. The proof assumes that "debt" is defined as the present value of nominally fixed outflows less the present value of nominally fixed inflows. This means that if a stream of riskless cash inflows is offset by a stream of riskless cash outflows, then there is no change in the amount of debt outstanding and therefore no change in leverage related costs or benefits. Since the offsetting riskless cash outflows eliminates all future net cash flows, the proceeds from the debt issue is the present value of the riskless cash inflows.

Proposition I: If a firm realizes the full value of its tax shields, the present value of a riskless stream of after tax cash flows is determined by discounting them at the after-tax riskless rate of interest.

Proof: Suppose a firm is going to receive a riskless cash flow of X_T T periods hence. The firm will have to pay taxes equal to the corporate tax rate, τ , times the cash flow so that its after-tax cash flow is $X_T(1 - \tau)$. Since the cash flow is riskless, the cash inflow is offset by a riskless cash outflow to prevent a change in the amount of debt outstanding. This is accomplished by constructing an equivalent loan. The equivalent loan is constructed as follows: in the initial period the firm borrows $\$B_0$. In the next period the firm repays $\$B_0(1 + r)$ to retire its debt. It finances this repayment from an interest tax shield of $\tau r \$B_0$ and a new loan of $\$B_1 = \$B_0(1 + r(1 - \tau))$. This rollover process continues in each period prior to period T . That is, in period t the firm repays its debt from the prior period, $B_{t-1}(1 + r)$ with its

interest tax shield $\$B_{t-1}\tau$ and a new loan of $\$B_t = B_{t-1}(1 + r(1 - \tau))$. In period T the loan from period $T - 1$ is repaid from the after-tax cash flow, $X_T(1 - \tau)$, and the tax shield $\$B_{T-1}\tau$. If the repayment at period T equals the after-tax cash flow, plus the tax shield in period T , then the only net cash flow from the riskless cash inflow of $X_T(1 - \tau)$ and the equivalent loan is the initial amount borrowed, $\$B_0$, and this, by definition, is the present value of $X_T(1 - \tau)$.

To find the initial amount borrowed, note that in period T

$$\begin{aligned} X_T(1 - \tau) &= B_{T-1}(1 + r) - B_{T-1}\tau r \\ &= B_{T-1}(1 + r(1 - \tau)) . \end{aligned}$$

Since the loan is refinanced each period,

$$B_t = B_{t-1}(1 + r(1 - \tau))$$

and, by recursive substitution,

$$X_T(1 - \tau) = B_0(1 + r(1 - \tau))^T .$$

The present value of $X_T(1 - \tau)$ is $\$B_0$,

$$PV(X_T(1 - \tau)) = B_0 = \frac{X_T(1 - \tau)}{(1 + r(1 - \tau))^T} .$$

The present value of a stream of riskless cash flows, $X_1(1 - \tau)$, $X_2(1 - \tau)$, . . . , $X_T(1 - \tau)$, is found similarly. Since present

values are additive,

$$PV(\underline{X}(1 - \tau)) = \sum_{t=1}^T PV(X_t(1 - \tau)) = \sum_{t=1}^T \frac{X_t(1 - \tau)}{1 + r(1 - \tau))^t}$$

where $\underline{X}(1 - \tau)$ is the stream of after-tax riskless cash flows.

The present value of riskless cash flows can also be calculated by discounting at the before tax interest rate and adding the present value of the interest tax shields. As the proof of the proposition above suggests, the appropriate interest tax shield in each period is the tax rate times the interest rate times the present value of project in the prior period.

Proposition II: If a firm realizes the full value of its tax shields, the present value of a riskless stream of after-tax cash flows is the sum of the after-tax cash flows discounted at the before tax interest rate plus the interest tax shields discounted at the before tax riskless interest rate. The interest tax shield in each period is the present value of the cash flows in the previous period times the product of the tax rate and the interest rate.

Proof: Suppose a firm is going to receive a riskless after-tax cash flow of $X_T(1 - \tau)$ T periods hence. Define V_t^T as the present value of $X_T(1 - \tau)$ in period t . According to the proposition, the present value, V_0^T , of $X_T(1 - \tau)$ is:

$$V_0^T = \frac{X_T(1 - \tau)}{(1 + r)^T} + \sum_{t=1}^T \frac{V_t^{T-1} \tau r}{(1 + r)^t} \quad (1)$$

From proposition I, the value of the cash flow grows at the after-tax

interest rate,

$$V_t^T = V_{t-1}^T (1 + r(1 - \tau)) . \quad (2)$$

Recursively substituting (2) into (1) and rearranging terms provides the expression

$$V_0^T = \frac{X_T(1 - \tau)}{(1 + r)^T} + \frac{\tau r V_0^T}{1 + r} \sum_{t=0}^{T-1} \left(\frac{1 + r(1 - \tau)}{1 + r} \right)^t \quad (3)$$

The summation on the right-hand-side of (3) equals

$$\sum_{t=0}^{T-1} \left(\frac{1 + r(1 - \tau)}{1 + r} \right)^t = \frac{1 - \left(\frac{1 + r(1 - \tau)}{1 + r} \right)^T}{\frac{\tau r}{1 + r}} \quad (4)$$

Substituting (4) into (3) and rearranging terms provides:

$$V_0^T = \frac{X_T(1 - \tau)}{(1 + r)^T} + V_0^T \left[1 - \left(\frac{1 + r(1 - \tau)}{1 + r} \right)^T \right] , \quad (5)$$

and thus:

$$V_0^T = \frac{X_T(1 - \tau)}{(1 + r(1 - \tau))^T}$$

which, according to proposition I, is the present value of $X_T(1 - \tau)$.

The present value of a stream of riskless after-tax cash flows, $X_1(1 - \tau)$, $X_2(1 - \tau)$, . . . , $X_T(1 - \tau)$, is

$$PV(\underline{X}(1 - \tau)) = \sum_{t=1}^T \frac{\underline{X}_t(1 - \tau)}{(1 + r)^t} + \sum_{t=1}^T \sum_{k=0}^t \frac{\tau r V_k^t}{(1 + r)^k}$$

where $\underline{X}(1 - \tau)$ denotes the stream of riskless after-tax cash flows.

Summary

There are at least two ways to calculate the present value of a stream of riskless cash flows. The first is to discount the after-tax cash flows by the after-tax interest rate. The second is to discount the after-tax cash flows by the before-tax rate and add the present value of the tax shields.

Pragmatically, the first method is much easier than the second since, to calculate the present value of the tax shields, the present value of the future cash flows (including the present value of the tax shields) must be calculated. Nevertheless, when cash flows are not riskless and thus are not offset entirely by riskless debt, the second method may provide a framework for applying techniques to value the risky interest tax shields.

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